Utility Functions and Optimisation

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Abstract

In this note we present mathematical formulations for the concepts of consumer preferences, utility functions and constrained optimisation. Appendices on linear, affined and monotonic transformations are included, as well as a simple proof that a Cobb-Douglass utility function is homothetic.

1. ***Introduction***

This section presents a linear model for the analysis of supply-demand functions for a single commodity and comparative static analysis. Basic techniques of Optimisation are discussed and the concept of derivative is introduced as the slope of a production function.

1.1. *Linear Models for Supply-Demand Functions*

A simple mathematical model for a supply-demand function consists of an equation for demand

$$q_d = \alpha_1 - \beta_1 p$$

where q_d is the amount demanded, α_1 is the *q*-intercept, β_1 is the slope of the demand line and *p* is the price of the commodity. The supply function is linear and represented by the equation

$$q_s = \alpha_2 + \beta_1 p$$

The point of equilibrium − where supply and demand coincide − is obtained from the solution to the linear system of equations

$$\begin{aligned} q_d &= \alpha_1 - \beta_1 p \\ q_s &= \alpha_2 + \beta_2 p \end{aligned}$$

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The solution to the system in terms of price is

$$p = \frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2}$$

The amount demanded at equilibrium can be expressed as a function

$$q^* = f(\alpha_1, \beta_1, \alpha_2, \beta_2) = \frac{\alpha_1\beta_2 + \beta_1\alpha_2}{\beta_1 + \beta_2}$$

The numbers $a,b,c,d \in Q_+$ are known as parameters. Changing the values of a,b,c or d moves the point of equilibrium. The trajectory of the system from point A to point B is not known within the framework of comparative statics. The equations involved in the trajectory do not lend themselves to analytical solutions.

1.2. **Numerical Example**.

Consider a demand function $q_d = f(p): R \rightarrow R$ with parameters $\alpha_1\beta_1 \in Z$ and equation

$$q_d = 100 - 200p$$

and a supply function

$$q_s = -300 + 150p$$

Equating q_s to q_d yields an equilibrium price of

$$100 - 200p = -300 + 150p \Rightarrow p^*$$

The variables in question are $q, p, (q_s, q_p)$. The parameters β_1 and β_2 indicate the slopes of the demand and supply lines, denoted by

$$\frac{\Delta q_d}{\Delta p} = -200, \qquad \frac{\Delta q_s}{\Delta p} = 150$$

1.3. **The Non-Linear Case**

Consider an economy with only two goods, x and y and related by the quadratic form [2]

$$ax^2 + y^2 = b$$

At maximum capacity, the economy produces x and y at the frontier, i.e. the graph of the quadratic form in $\mathbf{R}^2$. Isolating y in terms of x

$$y = \sqrt{b - ax^2}$$

The slope of the frontier curve is given by the limit

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

The first derivative of $y = f(x)$ is also a non linear function

$$\frac{dy}{dx} = -\frac{2x}{\sqrt{b - ax^2}}$$

An implicit expression for the derivative is obtained by substituting the quadratic form in the denominator by y

$$\frac{dy}{dx} = f(x, y) = -\frac{2x}{y}$$

1.4. **Optimisation of a single-variable function.**

For a function $y = f(x), f : R \to R$ the differentials of y and x are defined by

$$\Delta y = y_i - y_{i-1}, \qquad \Delta x = x_i - x_{i-1}$$

The differential Δy can be expressed in terms of Δx with the *Difference Quotient*

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \Rightarrow \Delta y = f(x + \Delta x) - f(x)$$

Taking the limit of the Difference Quotient as Δx tends to zero we obtain an expression for the first derivative of f

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \approx \frac{\Delta y}{\Delta x}$$

The first Derivative of *f* can be defined at a particular point $x_0 \in D_f$ as an ordered pair

$$(x_0, \frac{\Delta y}{\Delta x}\Big|_{x_0}) \quad \in R^2$$

1.4.1. *First-Order Condition.*

The optimum point $(x^*, f(x^*))$ occurs at a value in D_f where the first derivative is zero, i.e.

$$\frac{dy}{dx} = (x^*) = 0$$

1.4.2. *Second-Order Condition.*

The first-order condition is *necessary* but not *sufficient*: the point might be a maximum or a minimum of the function. The second-order condition is sufficient to establish a minimum or maximum point. At a minimum point

$$\frac{d^2y}{dx^2}(x^*) > 0$$

The condition implies that the function is upwards concave at the extremes. At a maximum

$$\frac{d^2y}{dx^2}(x^*) < 0$$

2. The Theory of Consumer Preferences and Utility

This section presents the fundamental assumptions leading to a consistent theory of consumer behaviour and the concept of utility. The assumptions in question include the rationality of the consumer's decision process, monotonicity and convexity.

2.1. **Axioms of Rational Behaviour**.

The analysis of consumer behaviour is based upon the *rationality* of economic agents. This assumption implies that economic agents are able to perform an economic calculation, thus identifying the best possible bundle of goods (i.e. the *optimal* bundle.) The following are the *axioms of rational behaviour* for economic agents. Consider an economy with two goods *x* and *y* and the bundles of goods

$$X = (x_1, y_1), Y = (x_2, y_2), Z = (x_3, y_3):$$

Completeness: Either one of the following relations exists

(1) $(x_1, y_1) \prec (x_2, y_2)$
(2) $(x_2, y_2) \prec (x_1, y_1)$
(3) $(x_1, y_1) \sim (x_2, y_2)$

Transitivity:

$$(x_1, y_1) \prec (x_2, y_2) \wedge (x_2, y_2) \prec (x_3, y_3) \Rightarrow (x_1, y_1) \prec (x_3, y_3)$$

Continuity:

$$(x_1 y_1) \prec (x_2, y_2) \Rightarrow \ (x_1 + \sum_{i=1}^{n} x_i, y_1 + \sum_{i=1}^{n} y_i) \prec \ (x_2 + \sum_{i=1}^{n} x_i, y_2 + \sum_{i=1}^{n} y_i, \quad)$$

:

$$\lim_{n \to \infty} \sum_{i=1}^{n} x_i = 0, \lim_{n \to \infty} \sum_{i=1}^{n} y_i = 0$$

2.2. **Utility Functions**.

The relation $\prec$ is an *order* relation. The concept of *utility* lets us order consumer preferences by assigning a real number to each bundle. A *Utility Function* is a real valued function $U : (x, y) \rightarrow R_+$ that satisfies

$$(x_{i+1}, y_{i+1}) \prec (x_i, y_i) \Rightarrow U(x_{i+1}, y_{i+1}) < U(x_i, y_i)$$

The ordering $(x_{i+1}, y_{i+1}) \prec (x_i , y_i)$ is preserved under any *Monotonic* transformation T

$$U(x_{i+1},y_{2+1})<U(x_i,y_i)\Rightarrow T[U(x_{i+1},y_{i+1})]< T[U(x_i,y_i)]$$

A utility function can be used to manifest preference towards a particular bundle of goods

$$U(x_1, x_2, \ldots, x_n)$$

Preference can also be manifested in terms of wealth *U(W)* or Consumption-Unemployment *U(C, u)*.

2.3. **Marginal Utility Analysis**.

Consider an utility function in R^3, $U : R_+ \supset u(x,y) \rightarrow (x,y) \subset R^2$. Let $\bar{u} \in R_+$ be a fixed level of utility. The *total derivative* of $\bar{U}$ is obtained by partial differentiation

$$\partial \bar{u} = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

Since $\bar{u}$ is a constant, $\partial \bar{u} = 0$. An expression for the first derivative $\frac{\partial y}{\partial x}$ at the utility level $\bar{u}$ results in the ratio of partial derivatives of U with respect to x and y

$$\frac{\partial U}{\partial x}\partial x + \frac{\partial U}{\partial y}\partial y = 0$$

$$\frac{\partial U}{\partial y}\partial y = -\frac{\partial U}{\partial x}\partial x$$

$$\left.\frac{\partial y}{\partial x}\right|_{\bar{u}} = -\frac{\partial U / \partial x}{\partial U / \partial y}$$

The Marginal rate of substitution, denoted by *MRS*, is simply the derivative evaluated at a fixed level of utility

$$\left.\frac{\partial y}{\partial x}\right|_{\bar{u}} =: MRS$$

2.4. **Assumptions on Utility Functions**.

Two basic assumptions rule the use of utility functions: *Monotonicity* and *Convexity.* Monotonicity entails the preservation of an order relation under monotonic transformations. The underlying economic presumption is a rational consumer who prefers *more* than *less* of a certain good. The assumption of convexity establishes the preference of consumers towards a balanced bundle of goods rather than an extreme one. Let A(x_1, y_1) and B(x_2, y_2) be bundles of goods on the same isoquant curve u. Defining a bundle C of goods as $C(\bar{x}, \bar{y})$, the following preference ordering holds

$$C\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \succ A(x_1, y_1), B(x_2, y_2)$$

[T.H. Barr,1997] A set $S \in \mathrm{R}^n$ is **convex** if for any two points $\mathrm{x}, \mathrm{y} \in S$, the points x + t (y – x), $t \in [0,1]$, all lie within S.

This definition means that, for any two points inside the set, any *linear* combination of these two points also lies within the set. The slope of the utility function is decreasing with a negative slope, i.e.

$$\frac{\partial y}{\partial x} < 0 \qquad\qquad \frac{\partial^2 y}{\partial x^2} > 0$$

An economic explanation for this property lies within the *law of diminishing returns*. While the consumption of good x increases as it becomes abundant, good y becomes scarce. The price of x goes down relative to the price of y, and the consumer is willing to sacrifice greater amounts of x in exchange for y.

2.5. Deriving an expression for $\frac{dy}{dx}$ at a fixed level of utility $\bar{u}$.

Consider an utility function U = U(x,y) with total differential

$$\partial \bar{u} = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

and three bundles X (x_1 ,y_1), Y (x_2 ,y_2), Z (x_3 ,y_3) $\in \bar{u}$ with $x_1 < x_2 < x_3$. Since $\bar{u}$ is convex

$$\frac{dy}{dx}\Big|_{\bar{u}}(X) > \frac{dy}{dx}\Big|_{\bar{u}}(Y) > \frac{dy}{dx}\Big|_{\bar{u}}(Z)$$

EXAMPLE. Consider an utility function. $U(x, y) = \sqrt{xy}$ and a fixed level $\bar{u}$. At this level of utility $\sqrt{xy} = \bar{u}$. Solving for y

$$y = \frac{\bar{u}^2}{x}$$

The first and second derivatives are the functions

$$\frac{dy}{dx} = -\frac{\bar{u}^2}{x^2} < 0 \qquad \frac{d^2y}{dx^2} = \frac{2\bar{u}^2}{x^3} > 0$$

2.5. Cobb-Douglass Utility Function

A *Cobb-Douglass* utility function is a two-variable real function of the form

$$u = u(x, y) = x^{\alpha} y^{\beta}$$

Taking the total differential of u by partial differentiation and taking into account that du equals zero at a fixed level $\bar{u}$.

$$\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = du$$
$$\alpha x^{\alpha-1} y^{\beta} dx + \beta x^{\alpha} y^{\beta-1} = 0$$
$$\left.\frac{dy}{dx}\right|_{\bar{u}} = \frac{-\alpha x^{\alpha-1} y^{\beta}}{\beta x^{\alpha} y^{\beta-1}}$$

Simplifying the above result yields

$$\left.\frac{dy}{dx}\right|_{\bar{u}} = -\frac{\alpha}{\beta}\frac{y}{x}$$

The function in question is **homothetic**, meaning that the function depends on the proportion of the variables, not on their magnitude.

2.7. **Leontief Utility Function**.

A *Leontief Utility function*, also known as Fixed Coefficient function is a discrete function of the form

$$u = u(x, y) = min(\alpha x, \beta y)$$

The Leontief function is homothetic and non-differentiable at $(\beta, \alpha) \in R^2$. This type of utility function characterizes consumer selection amongst *perfect complements*.

3. Optimisation and the Method of Lagrange Multipliers

The basic assumption of microeconomic analysis is the rationality of economic agents. A simple problem of optimisation with a single-valued real function was discussed in section 1.4. A more complex situation arises when more than two independent variables are present. The standard form of an optimisation problem, also called an *optimisation program* consists of an objective function subject to one or more restrictions

$$\begin{aligned} &\max f(x_1, \ldots, x_n) \\ &\text{s.t. } g_1(x_1, \ldots, x_n), \ldots., g_m(x_1, \ldots, x_n) \end{aligned}$$

The Method of *Lagrange Multipliers* is a powerful tool in solving optimisation programs. The method can also be extended to vector-valued functions.

3.1. **The Budget Set.**

Up to this point we have dealt with a selection process strictly based on consumer preferences. A more realistic approach would take into account the limitations on consumer choices posed by the availability of resources. Thus, even if the consumer prefers a certain bundle of goods, the choice of that bundle is contingent upon the consumer being able to *pay* for the cost of the goods.

Let Y be the income available to the consumer, p_x, p_y the price per unit of the goods *x* and *y* respectively [7] . The **Budget Set** is the set of all bundles (ordered pairs (x,y) $\in \mathbf{R}^2$) the consumer can get for his(her) income. In Cartesian space, the budget set is made up of all the points inside the convex region between the *x*-axis, the *y*-axis and the line $Y = p_x x + p_y y$

$$\{(x,y) \in \mathbf{R}^2 \mid p_x x + p_y \leq Y\} \subset \mathbf{R}^2$$

The budget set or **Budget Restriction** can also be represented using an implicit function

$$P_x x + p_y y - Y \leq 0$$

3.2. Analysis of the Budget Set when Y, p_x , p_y are fixed

Suppose income and prices are fixed, then they become *parameters* in the Budget Restriction equation. Differentiating both sides of the implicit representation of the Budget Set

$$(p_x dx + x dp_x) + (p_y dy + y dp_y) \leq 0$$

Since p_x and p_y are fixed, $dp_x = 0 = dp_y$, then

$$p_x dx + p_y dy \leqslant 0$$

Using cross-multiplication the above expression simplifies to

$$\frac{dy}{dx} \geqslant - \frac{p_x}{p_y}$$

In order to eliminate the existence of *border conditions* we must assume that all of Y is consumed, i.e. there are no savings. This assumption is essential to obtain a feasible solution to an optimisation program containing a budget restriction.

Appendix A

Linear Transformations

A function $T:R^n \rightarrow R^m$ is a linear transformation if $T(r\hat{v}) = rT(\hat{v}), r \in R$ and $T(\hat{u}+\hat{v}) = T(\hat{u}) + T(\hat{v})$. A linear transformation $T:R \rightarrow R$ is linear if and only if $T(x) = mx$ for some $x \in R$.

Appendix B

Monotonic Utilities

Consider an utility function $u(x_1,x_2)$ such that for bundles $(x_1^0, x_2^0), (x_1^1, x_2^1)$

$$(x_1^0, x_2^0) \succ (x_1^1, x_2^1) \Rightarrow u(x_1^0, x_2^0) < u(x_1^1, x_2^1)$$

Any linear transformation of u that does not alter the ordering of preferences is a **monotonic transformation**. Let T be such a transformation, then

$$T[u(x_1^0, x_2^0)] < T[u(x_1^1, x_2^1)]$$

Appendix C

Proof that a Cobb-Douglass utility function is homothetic

A Cobb-Douglass Utility function is a function of the form

$$u(x,y) = x^{\alpha} y^{\beta}$$

Since $\alpha + \beta = 1$, the above function can be expressed as

$$u(x,y) = x^{\alpha} y^{1-\alpha}$$

Let $T: (x,y) \rightarrow u(tx,ty),\ t \in R_+$ be a linear transformation. Applying the rule of assignment, or mapping, $u: (x,y) \rightarrow x^{\alpha} y^{\beta}$

$u(tx,ty) = (tx)^{\alpha} (ty)^{1-\beta}$	(by the mapping u)
$= t^{\alpha} x^{\alpha} t^{1-\alpha} y^{1-\alpha}$	(by the axioms of exponents)
$= t^{\alpha} t^{1-\alpha} x^{\alpha} y^{1-\alpha}$	(by associability of multiplication)
$= t\, x^{\alpha} y^{1-\alpha}$	(by the axioms of exponents)
$= t\, u(x,y)$	(by the axiom of substitution)

If $u(x_1,y_1) < u(x_2,y_2)$ then $u(tx_1,ty_1) < u(tx_2,ty_2)$, therefore $u(x,y)$ is homothetic.QED.